ISOPARAMETRIC FOLIATION AND YAU CONJECTURE ON THE FIRST EIGENVALUE, II

ZIZHOU TANG, YUQUAN XIE, AND WENJIAO YAN

Abstract. This is a continuation of [TY], which investigated the first eigenvalues of minimal isoparametric hypersurfaces with $g = 4$ distinct principal curvatures and focal submanifolds in unit spheres. For the focal submanifolds with $g = 6$, the present paper obtains estimates on all the eigenvalues, among others, giving an affirmative answer in one case to the problem posed in [TY], which may be regarded as a generalization of Yau's conjecture. In two of the four unsettled cases in [TY] for focal submanifolds M_1 of OT-FKM-type, we prove the first eigenvalues to be their dimensions, respectively.

1. Introduction

Let M^n be an n-dimensional closed connected Riemannian manifold and Δ be the Laplace-Beltrami operator acting on a C^∞ function f on M by $\Delta f = -\operatorname{div}(\nabla f)$, the negative of divergence of the gradient ∇f. It is well known that Δ is an elliptic operator and has a discrete spectrum

$$\{0 = \lambda_0(M) < \lambda_1(M) \leqslant \lambda_2(M) \leqslant \cdots \leqslant \lambda_k(M), \cdots, \uparrow \infty\}$$

with each eigenvalue occurs as many times as its multiplicity. As usual, we call $\lambda_1(M)$ the first eigenvalue of M. A well known conjecture of S.T.Yau states that

Yau conjecture ([Yau]): *The first eigenvalue of every (embedded) closed minimal hypersurface M^n in the unit sphere $S^{n+1}(1)$ is just n.*

Up to now, Yau's conjecture is still far from being solved. The most recent contribution to this problem is given by [TY]. They give an affirmative answer to Yau's conjecture for closed minimal isoparametric hypersurfaces M^n in $S^{n+1}(1)$.

By definition, an isoparametric hypersurface M^n in the unit sphere $S^{n+1}(1)$ is a hypersurface with constant principal curvatures. We denote by g the number of distinct principal curvatures, and m_1, m_2 their multiplicities (details will be discussed in the next section).

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In fact, for the minimal isoparametric hypersurfaces with $g = 6$, the proof of Yau's conjecture is just a simple combination of the results of [MOU], [Kot] with the classification theorems of [DN] and [Miy1], [Miy2]. An interesting problem naturally arises, is it possible to give a direct proof without using the classification theorems of Dorfmeister-Neher and Miyaoka (which states that all the isoparametric hypersurfaces with $g = 6$ in unit spheres are homogeneous)? As the first result of this paper, we provide a direct proof. Moreover, we obtain more information than that in [MOU], which only focused on the first eigenvalue of the minimal homogeneous hypersurfaces.

Theorem 1.1. Let M^{12} be a closed minimal isoparametric hypersurface in $S^{13}(1)$ with $g = 6$ and $(m_1, m_2) = (2, 2)$. Then

$$\lambda_1(M^{12}) = 12$$

with multiplicity 14. Furthermore, we have the inequality

$$\lambda_k(M^{12}) > \frac{3}{7}\,\lambda_k(S^{13}(1)), \qquad k = 1, 2, \cdots .$$

Other than the minimal isoparametric hypersurfaces, [TY] originally studied the first eigenvalues of the focal submanifolds of the isoparametric foliation in $S^{n+1}(1)$, which are in fact the minimal submanifolds in $S^{n+1}(1)$.

Theorem 1.3 in [TY]. *Let M_1 be the focal submanifold of an isoparametric hypersurface with $g = 4$ in $S^{n+1}(1)$. If $\dim M_1 \geqslant \frac{2}{3}n + 1$, then*

$$\lambda_1(M_1) = \dim M_1$$

with multiplicity $n + 2$. A similar conclusion holds for the other focal submanifold M_2.

As asserted in [TY], there are only four unsettled cases for the first eigenvalues of the focal submanifolds M_1 (*i.e.*, $f^{-1}(1)$, f is the restriction of the OT-FKM polynomial on the unit sphere) in the isoparametric foliation of OT-FKM type ($g = 4$). Namely, $(m_1, m_2) = (1, 1), (4, 3)$ (two cases: homogeneous and inhomogeneous) and $(5, 2)$. Unfortunately, their method is invalid for these cases. As the next aim of this paper, we consider M_1 with multiplicity pairs $(m_1, m_2) = (1, 1)$ and $(4, 3)$ (homogeneous), obtaining one of our main results as follows.

Theorem 1.2. For the focal submanifold M_1 of OT-FKM type in $S^5(1)$ with $(m_1, m_2) = (1, 1)$,

$$\lambda_1(M_1) = \dim M_1 = 3$$

with multiplicity 6; for the focal submanifold M_1 of homogeneous OT-FKM type in $S^{15}(1)$ with $(m_1, m_2) = (4, 3)$,

$$\lambda_1(M_1) = \dim M_1 = 10$$

with multiplicity 16.

Remark 1.1. As asserted in [TY], the first eigenvalue of the focal submanifold M_2 of OT-FKM type in $S^5(1)$ with $(m_1, m_2) = (1, 1)$ is equal to its dimension. As for the focal submanifold M_2 of homogeneous OT-FKM type in $S^{15}(1)$ with $(m_1, m_2) = (4, 3)$, its dimension satisfies the assumption of Theorem 1.3 in [TY], thus the first eigenvalue is equal to its dimension. By virtue of eigenfunctions constructed by Solomon on M_1 of OT-FKM type with $(m_1, m_2) = (5, 2)$, we see that the first eigenvalue is less than its dimension (*cf.* [Sol1]).

Notice that in their method calculating the first eigenvalues of the focal submanifolds, [TY] took average value of the gradient of the test functions at each pair of antipodal points. However, in the case $g = 6$, the average value is not accurate enough to meet our requirement. In this paper, by investigating the shape operators of the focal submanifolds, we obtain estimates on the first eigenvalus.

Theorem 1.3. *For the focal submanifolds of an isoparametric foliation with $g = 6$, we have*

(i) *when $(m_1, m_2) = (1, 1)$, the first eigenvalues of the focal submanifolds M_1 and M_2 in $S^7(1)$ satisfy*

$$3 \leqslant \lambda_1(M_1),\ \ \lambda_1(M_2) \leqslant \dim M_1 = \dim M_2 = 5.$$

(ii) *when $(m_1, m_2) = (2, 2)$, the k-th eigenvalues of the focal submanifolds M_1 and M_2 in $S^{13}(1)$ satisfy*

$$\lambda_k(S^{13}(1)) \leqslant (3 + \frac{99\sqrt{3}}{40\pi}) \cdot \lambda_k(M_1), \quad \lambda_k(S^{13}(1)) \leqslant (6 - \frac{117\sqrt{3}}{20\pi}) \cdot \lambda_k(M_2),$$

for $k = 1, 2, \cdots$. In particular,

$$\lambda_1(M_2) = \dim M_2 = 10 \tag{1}$$

with multiplicity 14.

Remark 1.2. In the case $g = 6$ and $(m_1, m_2) = (2, 2)$, we will distinguish M_1 from M_2 in Section 4, following the notations in [Miy2]. The equality (1) in Theorem 1.3 gives in this case an affirmative answer to the problem in [TY], which may be regarded as a generalization of Yau's conjecture. Unfortunately, we haven't got the accurate value of the first eigenvalue of M_1. Notice that M_1 and M_2 are not congruent in $S^{13}(1)$ (*cf.* [Miy2]). In fact, comparing the Ricci tensors by Gauss equation, one finds that M_1 and M_2 are not isometric. The problem of the determination of $\lambda_1(M_1)$ is still open!

2. **Preliminary**

An oriented hypersurface M^n in the unit sphere $S^{n+1}(1)$ with constant principal curvatures is called an isoparametric hypersurface (*cf.* [Car1], [Car2], [CR]). It is well known that a closed isoparametric hypersurface is an oriented, embedded hypersurface. Denote by ξ a unit normal vector field along M^n in $S^{n+1}(1)$, g the number of distinct principal curvatures of M, $\cot\theta_\alpha$ ($\alpha = 1, ..., g$; $0 < \theta_1 < \cdots < \theta_g < \pi$) the principal curvatures with respect to ξ and m_α the multiplicity of $\cot\theta_\alpha$. According to Münzner ([Mün]), the number g must be $1, 2, 3, 4$ or 6; $m_\alpha = m_{\alpha+2}$ (indices mod g) and $\theta_\alpha = \theta_1 + \frac{\alpha-1}{g}\pi$ ($\alpha = 1, ..., g$).

For isoparametric hypersurfaces in unit spheres with $g = 1, 2, 3$, Cartan classified them to be homogeneous (*cf.* [Car1], [Car2]); when $g = 6$, Abresch ([Abr]) showed that the multiplicity of each principal curvatures only takes values $m_1 = m_2 = 1$ or 2. Dorfmeister-Neher ([DN]) and Miyaoka ([Miy2]) proved the homogeneity of such hypersurfaces, respectively; for the most complicated case $g = 4$, Cecil-Chi-Jensen ([CCJ]), Immervoll ([Imm]) and Chi ([Chi]) proved a far reaching result that they are either homogeneous or of OT-FKM-type except possibly for the case $(m_1, m_2) = (7, 8)$.

A well known result of Cartan states that isoparametric hypersurfaces come as a family of parallel hypersurfaces. To be more specific, given an isoparametric hypersurface M^n in $S^{n+1}(1)$ and a smooth field ξ of unit normals to M, for each $x \in M$ and $\theta \in \mathbb{R}$, we can define $\phi_\theta : M^n \to S^{n+1}(1)$ by

$$\phi_\theta(x) = \cos\theta\ x + \sin\theta\ \xi(x).$$

Clearly, $\phi_\theta(x)$ is the point at an oriented distance θ to M along the normal geodesic through x. If $\theta \neq \theta_\alpha$ for any $\alpha = 1, ..., g$, ϕ_θ is a parallel hypersurface to M at an oriented distance θ, which we will denote by M_θ henceforward. If $\theta = \theta_\alpha$ for some $\alpha = 1, ..., g$, it is easy to find that for any vector X in the principal distributions $E_\alpha(x) = \{X \in T_xM \mid A_\xi X = \cot\theta_\alpha X\}$, where A_ξ is the shape operator with respect to ξ, $(\phi_\theta)_*X = (\cos\theta - \sin\theta\cot\theta_\alpha)X = \frac{sin(\theta_\alpha-\theta)}{sin\theta_\alpha}X = 0$. In other words, in case that $\cot\theta = \cot\theta_\alpha$ is a principal curvature of M, ϕ_θ is not an immersion, which is actually a *focal submanifold* of codimension $m_\alpha + 1$ in $S^{n+1}(1)$.

As asserted by Münzner, regardless of the number of distinct principal curvatures of M, there are only two distinct focal submanifolds in a parallel family of isoparametric hypersurfaces, and every isoparametric hypersurface is a tube of constant radius over each focal submanifold. Denote by M_1 the focal submanifold in $S^{n+1}(1)$ at an oriented distance θ_1 along ξ from M with codimension $m_1 + 1$, M_2 the focal submanifold in $S^{n+1}(1)$ at an oriented distance $\frac{\pi}{g} - \theta_1$ along $-\xi$ from M with codimension $m_2 + 1$. Another choice of the normal direction will lead to the exchange between the focal submanifolds M_1 and M_2. In virtue of Cartan's identity, one sees that both the focal submanifolds M_1 and M_2 are minimal in $S^{n+1}(1)$ (*cf.* [CR]).

3. Isoparametric hypersurfaces with $(g, m_1, m_2) = (6, 2, 2)$.

Let $\phi : M^n \to S^{n+1}(1)(\subset \mathbb{R}^{n+2})$ be a closed isoparametric hypersurface and again M_θ be the parallel hypersurface defined by $\phi_\theta : M^n \to S^{n+1}(1)$ $(-\pi < \theta < \pi, \cot\theta \neq \cot\theta_\alpha)$,

$$\phi_\theta(x) = \cos\theta\ x + \sin\theta\ \xi(x).$$

It is clear that for $X \in E_\alpha$,

$$(\phi_\theta)_* X = \frac{\sin(\theta_\alpha - \theta)}{\sin\theta_\alpha}\widetilde{X}, \tag{2}$$

where $\widetilde{X} \mathbin{/\!/} X$ as vectors in $\mathbb{R}^{n+2}$.

Following [TY], we will apply the theorem below to the case $V = S^{n+1}(1)$ and $W = M_1 \cup M_2$ and prove Theorem 1.1 by estimating the eigenvalue $\lambda_k(M^n)$ from below.

Theorem (Chavel and Feldman [CF]**, Ozawa** [Oza]**)** *Let V be a closed, connected Riemannian manifold and W a closed submanifold. For any sufficiently small $\varepsilon > 0$, set $W(\varepsilon) = \{x \in V :\ dist(x, W) < \varepsilon\}$. Let $\lambda_k^D(\varepsilon)$ $(k = 1, 2, ...)$ be the k-th eigenvalue on $V - W(\varepsilon)$ under the Dirichlet boundary condition. If* $\dim V \geqslant \dim W + 2$*, then*

$$\lim_{\varepsilon\to 0} \lambda_k^D(\varepsilon) = \lambda_{k-1}(V). \tag{3}$$

Proof of Theorem 1.1. In our case with $(g, m_1, m_2) = (6, 2, 2)$, denote by M^{12} a minimal isoparametric hypersurface. Clearly, $\theta_1 = \frac{\pi}{12}$. For sufficiently small $\varepsilon > 0$, set

$$M(\varepsilon) = \bigcup_{\theta\in[-\frac{\pi}{12}+\varepsilon,\ \frac{\pi}{12}-\varepsilon]} M_\theta,$$

which is a tube around M^{12}. According to the previous theorem,

$$\lim_{\varepsilon\to 0} \lambda_{k+1}^D(M(\varepsilon)) = \lambda_k(S^{13}(1)), \qquad k = 1, 2, \cdots. \tag{4}$$

Let $\left\{\widetilde{e}_{\alpha,i} \mid i = 1, 2,\ \alpha = 1, .., 6,\ \widetilde{e}_{\alpha,i} \in E_\alpha\right\}$ be a local orthonormal frame field on M. Then

$$\left\{\frac{\partial}{\partial\theta},\ e_{\alpha,i} \mid e_{\alpha,i} = \frac{\sin\theta_\alpha}{\sin(\theta_\alpha - \theta)}\widetilde{e}_{\alpha,i},\ i = 1, 2,\ \alpha = 1, .., 6,\ \theta \in [-\frac{\pi}{12} + \varepsilon,\ \frac{\pi}{12} - \varepsilon]\right\}$$

constitutes a local orthonormal frame field on $M(\varepsilon)$. From the formula (2), we derive the following equality up to a sign:

$$dM(\varepsilon) = 16\cos^2 2\theta \sin^2(\frac{\pi}{6} + 2\theta)\sin^2(\frac{\pi}{6} - 2\theta)d\theta dM \tag{5}$$

where $dM(\varepsilon)$ and dM are the volume elements of $M(\varepsilon)$ and M, respectively.

Again following [TY], let h be a nonnegative, increasing smooth function on $[0,\infty)$ satisfying $h=1$ on $[2,\infty)$ and $h=0$ on $[0,1]$. For sufficiently small $\eta>0$, let ψ_η be a nonnegative smooth function on $[\eta,\frac{\pi}{2}-\eta]$ such that

(*i*) $\psi_\eta(\eta)=\psi_\eta(\frac{\pi}{2}-\eta)=0$,

(*ii*) ψ_η is symmetric with respect to $x=\frac{\pi}{4}$,

(*iii*) $\psi_\eta(x)=h(\frac{x}{\eta})$ on $[\eta,\frac{\pi}{4}]$.

Let f_k $(k=0,1,...)$ be the k-th eigenfunctions on M which are orthogonal to each other with respect to the square integral inner product on M and $L_{k+1}=Span\{f_0,f_1,...,f_k\}$.

For each fixed $\theta\in[-\frac{\pi}{12}+\varepsilon,\ \frac{\pi}{12}-\varepsilon]$, denote $\pi=\pi_\theta=\phi_\theta^{-1}:M_\theta\to M$. Then any $\varphi\in L_{k+1}$ on M can give rise to a function $\Phi_\varepsilon:M(\varepsilon)\to\mathbb{R}$ by

$$\Phi_\varepsilon(x)=\psi_{3\varepsilon}(3(\frac{\pi}{12}-\theta))(\varphi\circ\pi)(x),$$

where θ is characterized by $x\in M_\theta$. It is easily seen that Φ_ε is a smooth function on $M(\varepsilon)$ satisfying the Dirichlet boundary condition and square integrable.

By the mini-max principle, we obtain:

$$\lambda^D_{k+1}(M(\varepsilon))\leqslant\sup_{\varphi\in L_{k+1}}\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2}. \tag{6}$$

In the following, we will concentrate on the calculation of $\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2}$. Observing that the normal geodesic starting from M is perpendicular to each M_θ, we obtain

$$\|\nabla\Phi_\varepsilon\|_2^2=\int_{M(\varepsilon)}9(\psi'_{3\varepsilon})^2\varphi(\pi)^2dM(\varepsilon)+\int_{M(\varepsilon)}\psi^2_{3\varepsilon}|\nabla\varphi(\pi)|^2dM(\varepsilon).$$

On the other hand, a simple calculation leads to

$$\begin{aligned}\|\Phi_\varepsilon\|_2^2 &= \int_{M(\varepsilon)}\psi^2_{3\varepsilon}(3(\frac{\pi}{12}-\theta))\varphi(\pi(x))^2dM(\varepsilon)\\ &= \int_M\int_{-\frac{\pi}{12}+\varepsilon}^{\frac{\pi}{12}-\varepsilon}16\cos^2 2\theta\sin^2(\frac{\pi}{6}+2\theta)\sin^2(\frac{\pi}{6}-2\theta)\psi^2_{3\varepsilon}(3(\frac{\pi}{12}-\theta))\varphi(\pi(x))^2d\theta dM\\ &= \frac{16}{3}\|\varphi\|_2^2\Big(\int_{3\varepsilon}^{\frac{\pi}{2}-3\varepsilon}\psi^2_{3\varepsilon}(x)\sin^2(\frac{2}{3}x)\cos^2(\frac{\pi}{6}-\frac{2}{3}x)\sin^2(\frac{\pi}{3}-\frac{2}{3}x)\ dx\Big).\end{aligned}$$

For the sake of convenience, let us decompose

$$\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2}=I(\varepsilon)+II(\varepsilon), \tag{7}$$

with

$$\begin{aligned}I(\varepsilon) &= \frac{\int_{M(\varepsilon)}9(\psi'_{3\varepsilon})^2\varphi(\pi)^2dM(\varepsilon)}{\int_{M(\varepsilon)}(\psi_{3\varepsilon})^2\varphi(\pi)^2\ dM(\varepsilon)}\\ &= \frac{9\int_{3\varepsilon}^{\frac{\pi}{2}-3\varepsilon}(\psi'_{3\varepsilon}(x))^2\sin^2(\frac{2}{3}x)\cos^2(\frac{\pi}{6}-\frac{2}{3}x)\sin^2(\frac{\pi}{3}-\frac{2}{3}x)\ dx}{\int_{3\varepsilon}^{\frac{\pi}{2}-3\varepsilon}\psi^2_{3\varepsilon}(x)\sin^2(\frac{2}{3}x)\cos^2(\frac{\pi}{6}-\frac{2}{3}x)\sin^2(\frac{\pi}{3}-\frac{2}{3}x)\ dx}\end{aligned} \tag{8}$$

and

$$II(\varepsilon)=\frac{\int_{M(\varepsilon)}\psi_{3\varepsilon}^2|\nabla\varphi(\pi)|^2dM(\varepsilon)}{\int_{M(\varepsilon)}\psi_{3\varepsilon}^2\varphi(\pi)^2dM(\varepsilon)}. \tag{9}$$

Firstly, as in [TY], we deduce without difficulty that

$$\lim_{\varepsilon\to 0}I(\varepsilon)=0. \tag{10}$$

Next, we turn to the estimate on $II(\varepsilon)$. Decompose $\nabla\varphi=Z_1+\cdots+Z_6\in E_1\oplus\cdots\oplus E_6$, and set $k_\alpha=\frac{sin(\theta_\alpha-\theta)}{sin\theta_\alpha}$ for $\alpha=1,...,6$. It follows obviously that

$$\begin{cases} |\nabla\varphi|^2 & =|Z_1|^2+\cdots+|Z_6|^2\\ |\nabla\varphi(\pi)|^2 & =\dfrac{1}{k_1^2}|Z_1|^2+\cdots+\dfrac{1}{k_6^2}|Z_6|^2. \end{cases} \tag{11}$$

Moreover, for $\alpha=1,...,6$, define

$$\begin{aligned} K_\alpha &:= 16\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}}\frac{\cos^2(2\theta)\sin^2(\frac{\pi}{6}+2\theta)\sin^2(\frac{\pi}{6}-2\theta)}{k_\alpha^2}d\theta\\ G &:= 32\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}}\cos^2(2\theta)\sin^2(\frac{\pi}{6}+2\theta)\sin^2(\frac{\pi}{6}-2\theta)\ d\theta. \end{aligned} \tag{12}$$

Let $K=\max_\alpha\{K_\alpha\}$. Then combining (7), (8), (9), (10), (11) with (12), we accomplish that

$$\lim_{\varepsilon\to 0}\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2}=\frac{\sum_\alpha K_\alpha\|Z_\alpha\|_2^2}{\|\varphi\|_2^2\cdot\frac{1}{2}G}\leqslant\frac{2K}{G}\cdot\frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2}. \tag{13}$$

Therefore, putting (4), (6) and (13) together, we obtain

$$\lambda_k(S^{13}(1))=\lim_{\varepsilon\to 0}\lambda_{k+1}^D(M(\varepsilon))\leqslant\lim_{\varepsilon\to 0}\sup_{\varphi\in L_{k+1}}\frac{\|\nabla\Phi_\varepsilon\|_2^2}{\|\Phi_\varepsilon\|_2^2}\leqslant\lambda_k(M^{12})\frac{2K}{G}. \tag{14}$$

Comparing the leftmost side with the rightmost side of (14), we find a sufficient condition to complete the proof of Theorem 1.1:

$$K<\frac{7}{6}G. \tag{15}$$

Since then, $\lambda_{15}(S^{13}(1))=28<\lambda_{15}(M^{12})\cdot\frac{7}{3}$, which implies immediately that $\lambda_{15}(M^{12})>12$. On the other hand, recall that 12 is an eigenvalue of M^{12} with multiplicity at least 14. Therefore, the first eigenvalue of M^{12} must be 12 with multiplicity 14.

We are left to verify the inequality (15). Observing that $K_1=K_6$, $K_2=K_5$ and $K_3=K_4$, we give the following straightforward verification.

(i)

$$\begin{aligned} K_1 &= 16\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \frac{\cos^2(2\theta)\sin^2(\frac{\pi}{6}+2\theta)\sin^2(\frac{\pi}{6}-2\theta)\sin^2\frac{\pi}{12}}{\sin^2(\frac{\pi}{12}-\theta)}\,d\theta \\ &= 16(2-\sqrt{3})(\frac{\pi}{64}+\frac{63\sqrt{3}}{1280}), \end{aligned}$$

while

$$G = 32\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \cos^2(2\theta)\sin^2(\frac{\pi}{6}+2\theta)\sin^2(\frac{\pi}{6}-2\theta)\ d\theta = \frac{\pi}{6}.$$

Therefore,

$$K_1 < \frac{7}{6}G.$$

(ii)

$$\begin{aligned} K_2 &= 16\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \frac{\cos^2(2\theta)\sin^2(\frac{\pi}{6}+2\theta)\sin^2(\frac{\pi}{6}-2\theta)\sin^2\frac{3}{12}\pi}{\sin^2(\frac{3}{12}\pi-\theta)}\,d\theta \\ &< 32\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \cos^2(2\theta)\sin^2(\frac{\pi}{6}+2\theta)\sin^2(\frac{\pi}{6}-2\theta)\ d\theta \\ &= G. \end{aligned}$$

(iii)

$$\begin{aligned} K_3 &= 16\int_{-\frac{\pi}{12}}^{\frac{\pi}{12}} \frac{\cos^2(2\theta)\sin^2(\frac{\pi}{6}+2\theta)\sin^2(\frac{\pi}{6}-2\theta)\sin^2(\frac{5}{12}\pi)}{\sin^2(\frac{5}{12}\pi-\theta)}\,d\theta \\ &< \frac{2+\sqrt{3}}{6}\cdot G \\ &< G. \end{aligned}$$

The proof of Theorem 1.1 is now complete.

□

4. Focal submanifolds with $g = 6$.

4.1. On the focal submanifolds M_1 and M_2 with $(g, m_1, m_2) = (6, 1, 1)$.

This subsection will be devoted to a proof of Theorem 1.3 (1).

Firstly, as mentioned before, the focal submanifolds are both minimal in unit spheres. It follows that $\lambda_1(M_i) \leqslant \dim M_i = 5, i = 1, 2$. Next, we will only prove $\lambda_1(M_1) \geqslant 3$, as the proof for M_2 is verbatim with obvious changes on index ranges.

Recall the Dorfmeister-Neher theorem ([DN]) which states that the isoparametric hypersurface in $S^7(1)$ with $(g, m_1, m_2) = (6, 1, 1)$ is homogeneous. Further, as asserted by [MO], a homogeneous hypersurface in $S^7(1)$ with $g = 6$ is the inverse image of the

Cartan hypersurface in $S^4(1)$ with $g = 3$ under the Hopf fiberation (for the eigenvalues of Cartan hypersurfaces, see [Sol2]); this correspondence exists between focal submanifolds of each hypersurface. Thus under the adjustment of the radius, we get the following Riemannian submersion with totally geodesic fibers:

$$\begin{array}{ccc} S^3(1) \hookrightarrow M_1 & \subset & S^7(1) \\ \downarrow & & \downarrow \\ S^2(\frac{\sqrt{3}}{2})/\mathbb{Z}_2 & \subset & S^4(\frac{1}{2}) \end{array} \tag{16}$$

where $S^2(\frac{\sqrt{3}}{2})/\mathbb{Z}_2 \subset S^4(\frac{1}{2})$ is the Veronese embedding of the real projective plane of constant Gaussian curvature $\frac{4}{3}$ into Euclidean sphere of radius $\frac{1}{2}$.

Next, let us recall some backgrounds for the Laplacians of a Riemannian submersion π with totally geodesic fibers: $F \hookrightarrow M \xrightarrow{\pi} B$. We denote the Laplacian of M by Δ^M. At any point $m \in M$, the vertical Laplacian Δ_v is defined to be

$$(\Delta_v f)(m) = ((\Delta^{F_m})(f|_{F_m}))(m),$$

where $F_m = \pi^{-1}(\pi(m))$ is the fiber of π through m and Δ^{F_m} the Laplace operator of the metric induced by M on F_m. The horizontal Laplacian is the difference operator

$$\Delta_h = \Delta^M - \Delta_v.$$

According to Theorem 3.6 in [BB], the Hilbert space $L^2(M)$ admits a Hilbert basis consisting of simultaneous eigenfunctions for Δ^M and Δ_v. Then we can find a function ϕ satisfying:

$$\begin{cases} \Delta^{M_1}\phi = \lambda_1(M_1)\phi \\ \Delta_v\phi \ = \ b\phi. \end{cases}$$

Since $\Delta_h\phi = (\lambda_1(M_1) - b)\phi$ and Δ_h is a non-negative operator, we have

$$b \leqslant \lambda_1(M_1) \leqslant 5. \tag{17}$$

On the other hand, concerning the relation $Spec(\Delta_v) \subset Spec(S^3(1)) = \{0, 3, 8, ...\}$, we claim that $b \geqslant 3$. Otherwise, suppose $b = 0$, then ϕ is the composition of the fiberation projection with an eigenfunction on the base space, such that

$$\lambda_1(M_1) \geqslant \lambda_1(S^2(\frac{\sqrt{3}}{2})/\mathbb{Z}_2) = 8 > 5,$$

contradicting (17).

Therefore, we arrive at

$$3 \leqslant \lambda_1(M_1) \leqslant 5. \tag{18}$$

4.2. **The first eigenvalue of the focal submanifold M_2 with $(g, m_1, m_2) = (6, 2, 2)$.**

Firstly, for sufficiently small $\varepsilon > 0$, we set

$$M_2(\varepsilon) := S^{n+1}(1) - B_\varepsilon(M_1) = \bigcup_{\theta\in[0,\frac{\pi}{6}-\varepsilon]} M_\theta$$

where $B_\varepsilon(M_1) = \{x \in S^{n+1}(1) \mid dist(x, M_1) < \varepsilon\}$, M_θ is the isoparametric hypersurface with an oriented distance θ from M_2. Notice that the notation M_θ here is different from that we used before.

Given $\theta \in (0, \frac{\pi}{6} - \varepsilon]$, let $\{e_{\alpha,i} \mid i = 1, 2,\ \alpha = 1, .., 6,\ e_{\alpha,i} \in E_\alpha\}$ be a local orthonormal frame field on M_θ and ξ be the unit normal field of M_θ towards M_2. After a parallel translation along the normal geodesic from any point $x \in M_\theta$ to the point $p = \phi_\theta(x) \in M_2$, (where $\phi_\theta : M_\theta \to M_2$ is the focal map), the image of ξ is normal to the focal submanifold M_2 at p, which will still be denoted by ξ; $e_{1,i}$ $(i = 1, 2)$ turn to be normal vectors on M_2, which we will denote by $\widetilde{e}_{1,i}$, while the others are still tangent vectors on M_2, which we will denote by $\{\widetilde{e}_{2,i}, \widetilde{e}_{3,i}, \widetilde{e}_{4,i}, \widetilde{e}_{5,i}, \widetilde{e}_{6,i}\}$. They are determined by x.

For any $X \in T_xM_\theta$, we can decompose it as $X = X_1 + \cdots + X_6 \in E_1 \oplus \cdots \oplus E_6$. Identifying the principal distribution $E_\alpha(x)$ $(\alpha = 2, \cdots, 6,\ x \in M_\theta)$ with its parallel translation at $p = \phi_\theta(x) \in M_2$. The shape operator A_ξ at p is given in terms of its eigenvectors $\widetilde{X}_\alpha$ (the parallel translation of $X_\alpha, \alpha = 2, \cdots, 6$) by (*cf.* [Mün])

$$\begin{aligned}
&A_\xi\widetilde{X}_2 = \cot(\theta_2-\theta_1)\widetilde{X}_2 = \sqrt{3}\widetilde{X}_2, \quad && A_\xi\widetilde{X}_3 = \cot(\theta_3-\theta_1)\widetilde{X}_3 = \frac{\sqrt{3}}{3}\widetilde{X}_3,\\
&A_\xi\widetilde{X}_4 = \cot(\theta_4-\theta_1)\widetilde{X}_4 = 0, && A_\xi\widetilde{X}_5 = \cot(\theta_5-\theta_1)\widetilde{X}_5 = -\frac{\sqrt{3}}{3}\widetilde{X}_5,\\
&A_\xi\widetilde{X}_6 = \cot(\theta_6-\theta_1)\widetilde{X}_6 = -\sqrt{3}\widetilde{X}_6.
\end{aligned} \tag{19}$$

Namely, $\widetilde{X}_2, \widetilde{X}_3, \widetilde{X}_4, \widetilde{X}_5, \widetilde{X}_6$ belong to the eigenspaces $E(\sqrt{3}), E(\frac{\sqrt{3}}{3}), E(0), E(-\frac{\sqrt{3}}{3}), E(-\sqrt{3})$ of A_ξ, respectively.

On the other hand, for any point $p \in M_2$, at a point $x \in {\phi_\theta}^{-1}(p)$, the first principal distribution $E_1(x)$ is projected to be 0 under $(\phi_\theta)_*$; for the others, we have

$$\begin{aligned}
(\phi_\theta)_*e_{\alpha,i} &= \frac{\sin(\theta_\alpha-\theta)}{\sin\theta_\alpha}\widetilde{e}_{\alpha,i} = \frac{\sin\frac{\alpha-1}{6}\pi}{\sin(\frac{\alpha-1}{6}\pi+\theta)}\widetilde{e}_{\alpha,i}\\
&:= \widetilde{k}_{\alpha-1}\widetilde{e}_{\alpha,i}, \qquad i = 1, 2,\ \alpha = 2, \cdots, 6.
\end{aligned}$$

Denote by $\{\theta_{\alpha,i} \mid \alpha = 1, \cdots, 6,\ i = 1, 2\}$ the dual frame of $e_{\alpha,i}$. We then conclude that (up to a sign)

$$dM_\theta = \prod_{j=1}^{2}\prod_{\alpha=2}^{6}\theta_{\alpha,j} \wedge \prod_{i=1}^{2}\theta_{1,i} = \frac{1}{(\widetilde{k}_1\cdots\widetilde{k}_5)^2}\phi_\theta^*(dM_2) \wedge \prod_{i=1}^{2}\theta_{1,i}. \tag{20}$$

Let h be the same function as in last section. For sufficiently small $\eta > 0$, define $\widetilde{\psi}_\eta$ to be a nonnegative smooth function on $[0, \frac{\pi}{2} - \eta]$ by

$$\widetilde{\psi}_\eta(x) := \begin{cases} 1, & x \in [0, \frac{\pi}{4}] \\ h(\frac{\frac{\pi}{2}-x}{\eta}), & x \in [\frac{\pi}{4}, \frac{\pi}{2} - \eta] \end{cases}$$

Let f_k $(k = 0, 1, ...)$ be the k-th eigenfunctions on M_2 which are orthogonal to each other with respect to the square integral inner product on M_2 and $L_{k+1} = Span\{f_0, f_1, ..., f_k\}$. Then any $\varphi \in L_{k+1}$ on M_2 can give rise to a function $\widetilde{\Phi}_\varepsilon : M_2(\varepsilon) \to \mathbb{R}$ by:

$$\widetilde{\Phi}_\varepsilon(x) = \widetilde{\psi}_{3\varepsilon}(3\theta)(\varphi \circ \phi_\theta)(x).$$

Evidently, $\widetilde{\Phi}_\varepsilon$ is a smooth function on $M_2(\varepsilon)$ satisfying the Dirichlet boundary condition and square integrable on $M_2(\varepsilon)$.

As in last section, the calculation of $\|\nabla\widetilde{\Phi}_\varepsilon\|_2^2$ is closely related to $|\nabla\varphi(\phi_\theta)|^2$. According to (19), in the tangent space of M_2 at p, we can decompose $\nabla\varphi$ as $\nabla\varphi = Z_1 + Z_2 + Z_3 + Z_4 + Z_5 \in E(\sqrt{3}) \oplus E(\frac{\sqrt{3}}{3}) \oplus E(0) \oplus E(-\frac{\sqrt{3}}{3}) \oplus E(-\sqrt{3})$. Subsequently,

$$\begin{cases} |\nabla\varphi|_p^2 = |Z_1|^2 + \cdots + |Z_5|^2 \\ |\nabla\varphi(\phi_\theta)|_x^2 = \widetilde{k}_1^2|Z_1|^2 + \cdots + \widetilde{k}_5^2|Z_5|^2 \end{cases} \tag{21}$$

In the following, we intend to investigate the variation of $|\nabla\varphi(\phi_\theta)|^2$ along with the point x in the fiber sphere at p. For this purpose, we recall that each integral submanifold of the curvature distributions corresponding to $\cot\theta_j = \cot(\theta + \frac{j-1}{6}\pi)$ is a totally geodesic submanifold in M_θ with constant sectional curvature $1 + \cot^2\theta_j$ (*cf.* for example, [CCJ]). In our case, we denote by $S^2(\sin\theta) \subset M_\theta$ the fiber sphere at p. Then a similar calculation as in Section 3 leads to

$$\begin{aligned} \lim_{\varepsilon\to 0}\|\nabla\widetilde{\Phi}_\varepsilon\|_2^2 &= \lim_{\varepsilon\to 0}\int_{M_2(\varepsilon)}(\widetilde{\psi}_{3\varepsilon}(3\theta))^2|\nabla(\varphi\circ\phi_\theta)|^2 dM_2(\varepsilon) \\ &= \int_0^{\frac{\pi}{6}}\Big(\int_{M_\theta}\frac{|\nabla(\varphi\circ\phi_\theta)|^2}{(\widetilde{k}_1\cdots\widetilde{k}_5)^2}\phi_\theta^*(dM_2)dS^2(\sin\theta)\Big)d\theta \end{aligned} \tag{22}$$

Given a point $p \in M_2 \subset S^{13}(1)$, with respect to a suitable tangent orthonormal basis $e_\alpha, e_{\bar\alpha}$ $(\alpha = 1, ..., 5)$ of T_pM_2, as asserted by Miyaoka in [Miy2], the shape operators A_ξ, A_ζ and $A_{\bar\zeta}$ with respect to the mutually orthogonal unit normals: ξ and two other unit normals, say ζ and $\bar\zeta$, of M_2 are expressed respectively by diagonal matrix

$$A_\xi = \begin{pmatrix} \sqrt{3}I & & & & \\ & \frac{1}{\sqrt{3}}I & & & \\ & & 0 & & \\ & & & -\frac{1}{\sqrt{3}}I & \\ & & & & -\sqrt{3}I \end{pmatrix} \tag{23}$$

and symmetric matrices:

$$A_{\bar{\zeta}}=\begin{pmatrix} 0 & -I & 0 & 0 & 0\\ -I & 0 & 0 & \frac{2}{\sqrt{3}}I & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & \frac{2}{\sqrt{3}}I & 0 & 0 & -I\\ 0 & 0 & 0 & -I & 0\end{pmatrix}\qquad A_{\zeta}=\begin{pmatrix} 0 & J & 0 & 0 & 0\\ -J & 0 & 0 & -\frac{2}{\sqrt{3}}J & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & \frac{2}{\sqrt{3}}J & 0 & 0 & J\\ 0 & 0 & 0 & -J & 0\end{pmatrix}\tag{24}$$

where

$$I=\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix}\qquad J=\begin{pmatrix} 0 & -1\\ 1 & 0\end{pmatrix}.\tag{25}$$

As a crucial step in our calculation, we set $\xi(t,s)=:\cos t\,\xi+\sin t\cos s\,\bar{\zeta}+\sin t\sin s\,\zeta$ $(0<t<\pi,\ 0\leqslant s\leqslant 2\pi)$ and the corresponding shape operator $A(t,s)=:A_{\xi(t,s)}$, thus

$$A(t,s)=\begin{pmatrix} \sqrt{3}\cos t\ I & -\sin t\ e^{-is} & 0 & 0 & 0\\ -\sin t\ e^{is} & \frac{1}{\sqrt{3}}\cos t\ I & 0 & \frac{2}{\sqrt{3}}\sin t\ e^{-is} & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & \frac{2}{\sqrt{3}}\sin t\ e^{-is} & 0 & -\frac{1}{\sqrt{3}}\cos t\ I & -\sin t\ e^{-is}\\ 0 & 0 & 0 & -\sin t\ e^{is} & -\sqrt{3}\cos t\ I\end{pmatrix},\tag{26}$$

where e^{is} is a matrix defined by $e^{is}=:\cos s\ I+\sin s\ J$. The eigenvalues of $A(t,s)$ are still $\sqrt{3},\frac{\sqrt{3}}{3},0,-\frac{\sqrt{3}}{3}$ and $-\sqrt{3}$, while the corresponding eigenspaces of $A(t,s)$ are spanned by eigenvectors as follows:

$E(\sqrt{3})=\mathrm{Span}\{\varepsilon_1,\varepsilon_{\bar{1}}\}$ with

$$\begin{aligned}\varepsilon_1 &= \frac{1}{2\sqrt{2(1-\cos t)}}\Big(\sin t(1+\cos t)(\cos 2s\ e_1-\sin 2s\ e_{\bar{1}})-\sqrt{3}\sin^2 t(\cos s\ e_2-\sin s\ e_{\bar{2}})\\ &\qquad -\sqrt{3}\sin t(1-\cos t)e_4+(1-\cos t)^2(\cos s\ e_5+\sin s\ e_{\bar{5}})\Big),\\ \varepsilon_{\bar{1}} &= \frac{1}{2\sqrt{2(1-\cos t)}}\Big(\sin t(1+\cos t)(\sin 2s\ e_1+\cos 2s\ e_{\bar{1}})-\sqrt{3}\sin^2 t(\sin s\ e_2+\cos s\ e_{\bar{2}})\\ &\qquad -\sqrt{3}\sin t(1-\cos t)e_{\bar{4}}+(1-\cos t)^2(-\sin s\ e_5+\cos s\ e_{\bar{5}})\Big),\end{aligned}$$

$E(\frac{1}{\sqrt{3}}) = \mathrm{Span}\{\varepsilon_2, \varepsilon_{\bar{2}}\}$ with

$$\begin{aligned}
\varepsilon_2 &= \frac{1}{2\sqrt{2(1+\cos t)}}\Big(-\sqrt{3}\sin t(1+\cos t)(\cos 2s\ e_1 - \sin 2s\ e_{\bar{1}}) \\
&\quad +(1+\cos t)(1-3\cos t)(\cos s\ e_2 - \sin s\ e_{\bar{2}}) \\
&\quad +\sin t(1+3\cos t)e_4 + \sqrt{3}\sin^2 t(\cos s\ e_5 + \sin s\ e_{\bar{5}})\Big), \\
\varepsilon_{\bar{2}} &= \frac{1}{2\sqrt{2(1+\cos t)}}\Big(-\sqrt{3}\sin t(1+\cos t)(\sin 2s\ e_1 + \cos 2s\ e_{\bar{1}}) \\
&\quad +(1+\cos t)(1-3\cos t)(\sin s\ e_2 + \cos s\ e_{\bar{2}}) \\
&\quad +\sin t(1+3\cos t)e_{\bar{4}} + \sqrt{3}\sin^2 t(-\sin s\ e_5 + \cos s\ e_{\bar{5}})\Big),
\end{aligned}$$

$E(0) = \mathrm{Spann}\{\varepsilon_3, \varepsilon_{\bar{3}}\}$ with

$$\varepsilon_3 = e_3, \quad \varepsilon_{\bar{3}} = e_{\bar{3}},$$

$E(-\frac{1}{\sqrt{3}}) = \mathrm{Span}\{\varepsilon_4, \varepsilon_{\bar{4}}\}$ with

$$\begin{aligned}
\varepsilon_4 &= \frac{1}{2\sqrt{2(1-\cos t)}}\Big(\sqrt{3}\sin t(1-\cos t)(\cos 2s\ e_1 - \sin 2s\ e_{\bar{1}}) \\
&\quad +(1-\cos t)(1+3\cos t)(\cos s\ e_2 - \sin s\ e_{\bar{2}}) \\
&\quad +\sin t(1-3\cos t)e_4 + \sqrt{3}\sin^2 t(\cos s\ e_5 + \sin s\ e_{\bar{5}})\Big), \\
\varepsilon_{\bar{4}} &= \frac{1}{2\sqrt{2(1-\cos t)}}\Big(\sqrt{3}\sin t(1-\cos t)(\sin 2s\ e_1 + \cos 2s\ e_{\bar{1}}) \\
&\quad +(1-\cos t)(1+3\cos t)(\sin s\ e_2 + \cos s\ e_{\bar{2}}) \\
&\quad +\sin t(1-3\cos t)e_{\bar{4}} + \sqrt{3}\sin^2 t(-\sin s\ e_5 + \cos s\ e_{\bar{5}})\Big),
\end{aligned}$$

$E(-\sqrt{3}) = \mathrm{Span}\{\varepsilon_5, \varepsilon_{\bar{5}}\}$ with

$$\begin{aligned}
\varepsilon_5 &= \frac{1}{2\sqrt{2(1+\cos t)}}\Big(-\sin t(1-\cos t)(\cos 2s\ e_1 - \sin 2s\ e_{\bar{1}}) - \sqrt{3}\sin^2 t(\cos s\ e_2 - \sin s\ e_{\bar{2}}) \\
&\quad +\sqrt{3}\sin t(1+\cos t)e_4 + (1+\cos t)^2(\cos s\ e_5 + \sin s\ e_{\bar{5}})\Big), \\
\varepsilon_{\bar{5}} &= \frac{1}{2\sqrt{2(1+\cos t)}}\Big(-\sin t(1-\cos t)(\sin 2s\ e_1 + \cos 2s\ e_{\bar{1}}) - \sqrt{3}\sin^2 t(\sin s\ e_2 + \cos s\ e_{\bar{2}}) \\
&\quad +\sqrt{3}\sin t(1+\cos t)e_{\bar{4}} + (1+\cos t)^2(-\sin s\ e_5 + \cos s\ e_{\bar{5}})\Big).
\end{aligned}$$

Now express $\nabla\varphi$ as

$$\nabla\varphi = \sum_{\alpha}(a_\alpha e_\alpha + a_{\bar{\alpha}} e_{\bar{\alpha}}) = \sum_{\alpha}(b_\alpha \varepsilon_\alpha + b_{\bar{\alpha}}\varepsilon_{\bar{\alpha}}),$$

where $a_\alpha = e_\alpha(\varphi)$, $a_{\bar{\alpha}} = e_{\bar{\alpha}}(\varphi)$, $b_\alpha = \varepsilon_\alpha(\varphi)$, $b_{\bar{\alpha}} = \varepsilon_{\bar{\alpha}}(\varphi)$. It follows that

$$|\nabla\varphi(\phi_\theta)|^2 = \sum_{\alpha=1}^{5} \widetilde{k}_\alpha^2 (b_\alpha^2 + b_{\bar{\alpha}}^2).$$

Further, a direct calculation leads to

$$\begin{aligned}
b_1^2+b_{\bar 1}^2 &= I(t,s)+\frac{1}{8(1-\cos t)}\Big(\sin^2 t(1+\cos t)^2(a_1^2+a_{\bar 1}^2)+3\sin^4 t(a_2^2+a_{\bar 2}^2)\\
&\qquad +3\sin^2 t(1-\cos t)^2(a_4^2+a_{\bar 4}^2)+(1-\cos t)^4(a_5^2+a_{\bar 5}^2)\Big),\\
b_2^2+b_{\bar 2}^2 &= II(t,s)+\frac{1}{8(1+\cos t)}\Big(3\sin^2 t(1+\cos t)^2(a_1^2+a_{\bar 1}^2)+(1+\cos t)^2(1-3\cos t)^2(a_2^2+a_{\bar 2}^2)\\
&\qquad +\sin^2 t(1+3\cos t)^2(a_4^2+a_{\bar 4}^2)+3\sin^4 t(a_5^2+a_{\bar 5}^2)\Big),\\
b_4^2+b_{\bar 4}^2 &= IV(t,s)+\frac{1}{8(1-\cos t)}\Big(3\sin^2 t(1-\cos t)^2(a_1^2+a_{\bar 1}^2)+(1-\cos t)^2(1+3\cos t)^2(a_2^2+a_{\bar 2}^2)\\
&\qquad +\sin^2 t(1-3\cos t)^2(a_4^2+a_{\bar 4}^2)+3\sin^4 t(a_5^2+a_{\bar 5}^2)\Big),\\
b_5^2+b_{\bar 5}^2 &= V(t,s)+\frac{1}{8(1+\cos t)}\Big(\sin^2 t(1-\cos t)^2(a_1^2+a_{\bar 1}^2)+3\sin^4 t(a_2^2+a_{\bar 2}^2)\\
&\qquad +3\sin^2 t(1+\cos t)^2(a_4^2+a_{\bar 4}^2)+(1+\cos t)^4(a_5^2+a_{\bar 5}^2)\Big),
\end{aligned}$$

where $I(t,s)$, $II(t,s)$, $IV(t,s)$ and $V(t,s)$ are those items containing linear combinations of $\cos 2s\cos s$, $\cos 2s\sin s$, $\sin 2s\cos s$ and $\sin 2s\sin s$, whose integrals over $s\in[0,2\pi]$ vanish.

Transform $\int_0^\pi |\nabla\varphi\circ\phi_\theta|^2\sin t\, dt$ to be

$$\int_0^\pi |\nabla\varphi\circ\phi_\theta|^2\sin t\, dt=(a_1^2+a_{\bar 1}^2)A_1+(a_2^2+a_{\bar 2}^2)A_2+(a_3^2+a_{\bar 3}^2)A_3+(a_4^2+a_{\bar 4}^2)A_4+(a_5^2+a_{\bar 5}^2)A_5,$$

for some $A_1,\cdots,A_5$. It is not difficult to find that

$$A_1=A_2=A_4=A_5=\frac12(\widetilde{k}_1^2+\widetilde{k}_2^2+\widetilde{k}_4^2+\widetilde{k}_5^2).$$

Then we finally arrive at an estimate of $\|\nabla\widetilde{\Phi_\varepsilon}\|_2^2$ in (22):

$$\begin{aligned}
\lim_{\varepsilon\to 0}\|\nabla\widetilde{\Phi_\varepsilon}\|_2^2 &= \int_0^{\frac{\pi}{6}}\int_{M_2}\frac{\sin^2\theta}{(\widetilde{k}_1\cdots\widetilde{k}_5)^2}\cdot 2\pi\cdot\int_0^\pi|\nabla\varphi\circ\phi_\theta|^2\sin t\, dtdM_2d\theta\\
&= \int_0^{\frac{\pi}{6}}\int_{M_2}\frac{\sin^2\theta}{(\widetilde{k}_1\cdots\widetilde{k}_5)^2}\cdot 2\pi\cdot\Big(\frac12(\widetilde{k}_1^2+\widetilde{k}_2^2+\widetilde{k}_4^2+\widetilde{k}_5^2)(a_1^2+a_{\bar 1}^2+a_2^2+a_{\bar 2}^2\\
&\qquad +a_4^2+a_{\bar 4}^2+a_5^2+a_{\bar 5}^2)+2\widetilde{k}_3^2(a_3^2+a_{\bar 3}^2)\Big)dM_2d\theta\\
&< \int_0^{\frac{\pi}{6}}\int_{M_2}\frac{\sin^2\theta}{(\widetilde{k}_1\cdots\widetilde{k}_5)^2}\cdot 2\pi\cdot\Big(\frac12(\widetilde{k}_1^2+\widetilde{k}_2^2+\widetilde{k}_4^2+\widetilde{k}_5^2)\sum_{\alpha=1}^5(a_\alpha^2+a_{\bar\alpha}^2)\Big)dM_2d\theta\\
&= \int_0^{\frac{\pi}{6}}\frac49\sin^2\theta\cos^2\theta(\frac14-\cos^2 2\theta)^2\Big(\frac{2-\cos 2\theta}{(\frac12-\cos 2\theta)^2}+\frac{3(2+\cos 2\theta)}{(\frac12+\cos 2\theta)^2}\Big)d\theta\cdot 2\pi\cdot\|\nabla\varphi\|_2^2\\
&= (\frac{\pi}{18}-\frac{13\sqrt3}{240})\pi\cdot\|\nabla\varphi\|_2^2
\end{aligned}$$

Combining with

$$\begin{aligned}\lim_{\varepsilon\to 0}\|\widetilde{\Phi_\varepsilon}\|_2^2 &= \int_0^{\frac{\pi}{6}}\int_{M_2}\frac{\sin^2\theta}{(\widetilde{k}_1\cdots\widetilde{k}_5)^2}\int_{M_2}\int_{S^2(\sin\theta)}\varphi(\phi_\theta)^2\ dS^2dM_2d\theta\\ &= 4\pi\cdot\|\varphi\|_2^2\cdot\int_0^{\frac{\pi}{6}}\frac{\sin^2\theta}{(\widetilde{k}_1\cdots\widetilde{k}_5)^2}\ d\theta\\ &= \frac{16\pi}{9}\cdot\|\varphi\|_2^2\cdot\int_0^{\frac{\pi}{6}}\sin^2 2\theta(\frac{1}{4}-\cos^2 2\theta)^2\ d\theta\\ &= \frac{\pi^2}{108}\cdot\|\varphi\|_2^2\end{aligned}$$

we conclude that

$$\lim_{\varepsilon\to 0}\frac{\|\nabla\widetilde{\Phi_\varepsilon}\|_2^2}{\|\widetilde{\Phi_\varepsilon}\|_2^2} \quad < \quad (6-\frac{117\sqrt{3}}{20\pi})\cdot\frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2}.$$

Similarly as the arguments in last section, we derive that

$$\lambda_k(S^{13}(1))\leqslant(6-\frac{117\sqrt{3}}{20\pi})\ \lambda_k(M_2)<\frac{14}{5}\ \lambda_k(M_2), \tag{27}$$

as $(6-\dfrac{117\sqrt{3}}{20\pi})\approx 2.774726$. Taking $k=15$, the inequality turns to

$$\lambda_{15}(M_2)>10.$$

At last, recalling Lemma 3.1 in [TY] which yields that the dimension 10 of M_2 is an eigenvalue of M_2 with multiplicity at least 14, we arrive at

$$\lambda_1(M_2)=\dim M_2=10 \text{ with multiplicity } 14,$$

as required. The proof of Theorem 1.3 (ii) for M_2 is now complete.

4.3. **On the focal submanifold M_1 with $(g,m_1,m_2)=(6,2,2)$.**

In this subsection, we still use the previous method to define similar neighborhood $M_1(\varepsilon)$ of M_1 and the test function $\widetilde{\Phi_\varepsilon}$. In the following, we will just list difference in the crucial step.

Given a point $p\in M_1\subset S^{13}(1)$, with respect to a suitable tangent orthonormal basis $e_\alpha, e_{\bar\alpha}$ $(\alpha=1,...,5)$ of T_pM_1, as asserted by Miyaoka in [Miy2], the shape operators A_ξ, A_ζ and $A_{\bar\zeta}$ with respect to the mutually orthogonal unit normals: ξ and two other unit normals, say ζ and $\bar\zeta$, of M_2 are expressed respectively by symmetric matrices:

$$A_\xi=\begin{pmatrix}\sqrt{3}I&&&&\\&\frac{1}{\sqrt{3}}I&&&\\&&0&&\\&&&-\frac{1}{\sqrt{3}}I&\\&&&&-\sqrt{3}I\end{pmatrix} \tag{28}$$

(29)

$$A_{\bar{\zeta}} = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3}I \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}}I & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}}I & 0 & 0 & 0 \\ -\sqrt{3}I & 0 & 0 & 0 & 0 \end{pmatrix} \qquad A_{\zeta} = \begin{pmatrix} 0 & 0 & 0 & 0 & \sqrt{3}J \\ 0 & 0 & 0 & \frac{1}{\sqrt{3}}J & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{3}}J & 0 & 0 & 0 \\ -\sqrt{3}J & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{30}$$

For the unit normal vector $\xi(t,s) =: \cos t\ \xi + \sin t \cos s\ \zeta + \sin t \sin s\ \bar{\zeta}$ $(0 < t < \pi,\ 0 \leqslant s \leqslant 2\pi)$, the corresponding shape operator $A(t,s) =: A_{\xi(t,s)}$ is given by

(31)

$$A(t,s) = \begin{pmatrix} \sqrt{3}\cos t\ I & 0 & 0 & 0 & \sqrt{3}\sin t\ e^{i(\frac{\pi}{2}-s)} \\ 0 & \frac{1}{\sqrt{3}}\cos t\ I & 0 & \frac{1}{\sqrt{3}}\sin t\ e^{i(\frac{\pi}{2}-s)} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{3}}\sin t\ e^{-i(\frac{\pi}{2}-s)} & 0 & -\frac{1}{\sqrt{3}}\cos t\ I & 0 \\ \sqrt{3}\sin t\ e^{-i(\frac{\pi}{2}-s)} & 0 & 0 & 0 & -\sqrt{3}\cos t\ I \end{pmatrix}.$$

The eigenvalues of $A(t,s)$ are still $\sqrt{3}, \frac{\sqrt{3}}{3}, 0, -\frac{\sqrt{3}}{3}$ and $-\sqrt{3}$, while the corresponding eigenspaces of $A(t,s)$ are spanned by eigenvectors as follows:

$E(\sqrt{3}) = \mathrm{Span}\{\varepsilon_1, \varepsilon_{\bar{1}}\}$ with

$$\begin{aligned} \varepsilon_1 &= \frac{\sin t \sin s}{\sqrt{2(1-\cos t)}} e_1 + \frac{\sin t \cos s}{\sqrt{2(1-\cos t)}} e_{\bar{1}} + \sqrt{\frac{1-\cos t}{2}} e_5 \\ \varepsilon_{\bar{1}} &= -\frac{\sin t \cos s}{\sqrt{2(1-\cos t)}} e_1 + \frac{\sin t \sin s}{\sqrt{2(1-\cos t)}} e_{\bar{1}} + \sqrt{\frac{1-\cos t}{2}} e_{\bar{5}}, \end{aligned}$$

$E(\frac{1}{\sqrt{3}}) = \mathrm{Span}\{\varepsilon_2, \varepsilon_{\bar{2}}\}$ with

$$\begin{aligned} \varepsilon_2 &= \frac{\sin t \sin s}{\sqrt{2(1-\cos t)}} e_2 + \frac{\sin t \cos s}{\sqrt{2(1-\cos t)}} e_{\bar{2}} + \sqrt{\frac{1-\cos t}{2}} e_4 \\ \varepsilon_{\bar{2}} &= -\frac{\sin t \cos s}{\sqrt{2(1-\cos t)}} e_2 + \frac{\sin t \sin s}{\sqrt{2(1-\cos t)}} e_{\bar{2}} + \sqrt{\frac{1-\cos t}{2}} e_{\bar{4}}, \end{aligned}$$

$E(0) = \mathrm{Spann}\{\varepsilon_3, \varepsilon_{\bar{3}}\}$ with

$$\varepsilon_3 = e_3, \quad \varepsilon_{\bar{3}} = e_{\bar{3}},$$

$E(-\frac{1}{\sqrt{3}}) = \mathrm{Span}\{\varepsilon_4, \varepsilon_{\bar{4}}\}$ with

$$\begin{aligned} \varepsilon_4 &= \frac{\sin t \sin s}{\sqrt{2(1+\cos t)}} e_2 + \frac{\sin t \cos s}{\sqrt{2(1+\cos t)}} e_{\bar{2}} - \sqrt{\frac{1+\cos t}{2}} e_4 \\ \varepsilon_{\bar{4}} &= -\frac{\sin t \cos s}{\sqrt{2(1+\cos t)}} e_2 + \frac{\sin t \sin s}{\sqrt{2(1+\cos t)}} e_{\bar{2}} - \sqrt{\frac{1+\cos t}{2}} e_{\bar{4}}, \end{aligned}$$

$E(-\sqrt{3}) = \mathrm{Span}\{\varepsilon_5, \varepsilon_{\bar{5}}\}$ with

$$\begin{aligned} \varepsilon_5 &= \frac{\sin t \sin s}{\sqrt{2(1+\cos t)}} e_1 + \frac{\sin t \cos s}{\sqrt{2(1+\cos t)}} e_{\bar{1}} - \sqrt{\frac{1+\cos t}{2}} e_5 \\ \varepsilon_{\bar{5}} &= -\frac{\sin t \cos s}{\sqrt{2(1+\cos t)}} e_1 + \frac{\sin t \sin s}{\sqrt{2(1+\cos t)}} e_{\bar{1}} - \sqrt{\frac{1+\cos t}{2}} e_{\bar{5}}, \end{aligned}$$

In an analogous way with that in last subsection, we obtain

$$\begin{aligned} \lim_{\varepsilon\to 0} \|\nabla \widetilde{\Phi_\varepsilon}\|_2^2 &= \int_0^{\frac{\pi}{6}} \int_{M_1} \frac{\sin^2\theta}{(\widetilde{k}_1\cdots\widetilde{k}_5)^2} \cdot 2\pi \cdot \int_0^{\pi} |\nabla\varphi \circ \phi_\theta|^2 \sin t \ dt dM_1 d\theta \\ &= 4\pi \int_{M_1} \Big((\frac{\pi}{36} - \frac{11\sqrt{3}}{240})(a_3^2 + a_{\bar{3}}^2) + (\frac{\pi}{144} + \frac{11\sqrt{3}}{1920})(a_1^2 + a_{\bar{1}}^2 + a_5^2 + a_{\bar{5}}^2) \\ &\qquad + (\frac{\pi}{48} - \frac{21\sqrt{3}}{640})(a_2^2 + a_{\bar{2}}^2 + a_4^2 + a_{\bar{4}}^2) \Big) \ dM_1 \\ &< 4\pi \int_0^{\frac{\pi}{6}} \frac{\sin^2\theta}{(\widetilde{k}_1\cdots\widetilde{k}_5)^2} \cdot \frac{\widetilde{k}_1^2 + \widetilde{k}_5^2}{2} \ d\theta \cdot \|\nabla\varphi\|_2^2 \\ &= (\frac{\pi^2}{36} + \frac{11\sqrt{3}}{480}\pi) \cdot \|\nabla\varphi\|_2^2 \end{aligned}$$

Combining with $\lim_{\varepsilon\to 0} \|\widetilde{\Phi_\varepsilon}\|_2^2 = \frac{\pi^2}{108} \cdot \|\varphi\|_2^2$, we eventually arrive at

$$\lambda_k(S^{13}(1)) \leqslant \lambda_k(M_1) \cdot \frac{\frac{\pi^2}{36} + \frac{11\sqrt{3}}{480}\pi}{\frac{\pi^2}{108}} = \lambda_k(M_1) \cdot (3 + \frac{99\sqrt{3}}{40\pi}),$$

as required. This completes the proof of Theorem 1.3 (ii) for M_1.

5. **Focal submanifolds with $g = 4$.**

We begin this section with a short review of the isoparametric hypersurfaces of OT-FKM-type. For a symmetric Clifford system $\{P_0, \cdots, P_m\}$ on $\mathbb{R}^{2l}$, *i.e.*, P_i's are

symmetric matrices satisfying $P_iP_j + P_jP_i = 2\delta_{ij}I_{2l}$, Ferus, Karcher and Münzner ([FKM]) constructed a polynomial F on $\mathbb{R}^{2l}$:

$$\begin{aligned} F: \quad & \mathbb{R}^{2l} \to \mathbb{R} \\ F(x) &= |x|^4 - 2\sum_{i=0}^{m} \langle P_i x, x\rangle^2. \end{aligned} \tag{32}$$

It turns out that each level hypersurface of $f = F|_{S^{2l-1}}$, *i.e.*, the preimage of some regular value of f, is an isoparametric hypersurface with four distinct constant principal curvatures. Choosing $\xi = \frac{\nabla f}{|\nabla f|}$, we have $M_1 = f^{-1}(1)$, $M_2 = f^{-1}(-1)$, which have codimensions $m_1 + 1$ and $m_2 + 1$ in $S^{n+1}(1)$, respectively. The multiplicity pairs $(m_1, m_2) = (m, l - m - 1)$, provided $m > 0$ and $l - m - 1 > 0$, where $l = k\delta(m)$ $(k = 1, 2, 3, ...)$ and $\delta(m)$ is the dimension of an irreducible module of the Clifford algebra $\mathcal{C}_{m-1}$ on $\mathbb{R}^l$.

5.1. **On the focal submanifold M_1 with $(g, m_1, m_2) = (4, 1, 1)$.**

As mentioned before, the isoparametric foliation with $(g, m_1, m_2) = (4, 1, 1)$ can be expressed in form of OT-FKM-type. In fact, by an orthogonal transformation, we can always choose the Clifford matrices P_0, P_1 to be

$$P_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, P_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then the focal submanifold M_1 is expressed as

$$M_1 =: \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \langle x, y\rangle = 0, |x| = |y| = \frac{1}{\sqrt{2}}\}.$$

In order to investigate M_1, we define a two-fold covering as below, regarding S^3 as the group of unit vectors in $\mathbb{H}$ of quaternions:

$$\begin{aligned} \sigma : S^3 \quad &\to \quad M_1 \subset \mathbb{R}^6 \\ a \quad &\mapsto \quad \frac{1}{\sqrt{2}}(aj\bar{a}, ak\bar{a}) \end{aligned}$$

where i, j, k are basis elements satisfying $i^2 = j^2 = k^2 = -1$ and $ij = k$.

Let us equip S^3 with the induced metric by σ. To be more specific, at any point $a \in S^3$, we can choose a basis of T_aS^3 as $e_1 =: ai$, $e_2 =: aj$, $e_3 =: ak$, whose images under the tangent map $\sigma_*(X) = \frac{1}{\sqrt{2}}(aj\bar{X} + Xj\bar{a}, ak\bar{X} + Xk\bar{a})$ $(X \in T_aS^3)$ are

$$\sigma_*(e_1) = (\sqrt{2}ak\bar{a}, -\sqrt{2}aj\bar{a}),\ \sigma_*(e_2) = (0, \sqrt{2}ai\bar{a}),\ \sigma_*(e_3) = (-\sqrt{2}ai\bar{a}, 0).$$

Subsequently, the metric matrix is

$$\Big(\langle e_p, e_q\rangle\Big) = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Therefore, S^3 with the induced metric is a Berger sphere, say S^3_B, and M_1 is isometric to the $\mathbb{Z}_2$-quotient $S^3_B/\mathbb{Z}_2$ by identifying its antipodal points. Actually, identifying $\mathbb{C}\times\mathbb{C}$ with $\mathbb{H}$ by $(z,w)\to z+jw$, we have the Hopf fiberation $S^3_B\to S^2$ defined by $(z,w)\mapsto(|z|^2-|w|^2, 2z\bar{w})$, which gives rise to the following Riemannian submersion with totally geodesic fibers and with the vertical space spanned by e_1:

$$\begin{array}{ccc} S^1(1)\hookrightarrow & S^3_B/\mathbb{Z}_2 & \cong M_1 \\ & \downarrow & \\ & S^2(\frac{\sqrt{2}}{2}) & \end{array} \tag{33}$$

Comparing with the Riemannian submersion with totally geodesic fibers

$$\begin{array}{cc} S^1(\dfrac{\sqrt{2}}{2})\hookrightarrow & S^3(\sqrt{2})/\mathbb{Z}_2 \\ & \downarrow\pi \\ & S^2(\frac{\sqrt{2}}{2}) \end{array} \tag{34}$$

where $S^3(\sqrt{2})$ is the standard sphere with radius $\sqrt{2}$, we can calculate the first eigenvalue of $M_1\cong S^3_B/\mathbb{Z}_2$ in the following steps.

Firstly, given a Riemannian submersion $\pi:(M,g)\to B$ with totally geodesic fibers, for each $t>0$, there is a unique Riemannian metric g_t on M, such that for any $m\in M$,

(i) $g_t|_{V_mM\times H_mM}=0$;
(ii) $g_t|_{V_mM}=t^2g|_{V_mM}$;
(iii) $g_t|_{H_mM}=g|_{H_mM}$.

We denote by M_{g_t} the Riemannian manifold (M,g_t) and by Δ^M_t its Laplacian. It is clear that $\Delta^M_t=t^{-2}\Delta_v+\Delta_h$ (*cf.* [BB]). Thus, a common eigenfunction of Δ_v and Δ_h is an eigenfunction of Δ^M_t.

In contrast with our case, we see that $M=S^3(\sqrt{2})/\mathbb{Z}_2$ and $M_{g_t}=S^3_B/\mathbb{Z}_2$ with $t=\sqrt{2}$.

Secondly, denote the spectrum of the Riemannian manifold M by $\{(\mu_k,n_k)\mid 0=\mu_0<\mu_1<\cdots<\mu_k<\cdots\uparrow\infty;\mu_k$ is an eigenvalue, n_k is the multiplicity of $\mu_k\}$. For the convenience, we list the well known spectrums of $S^1(1), S^1(\frac{\sqrt{2}}{2}), S^2(\frac{\sqrt{2}}{2}), S^3(\sqrt{2})/\mathbb{Z}_2$ in Table 1.

Finally, let Δ_h, Δ_v be the corresponding horizontal and vertical Laplacians in the Riemannian submersion (34). From Theorem 3.6 in [BB], it follows that for any $\lambda\in Spec(S^3(\sqrt{2})/\mathbb{Z}_2)$, there exist nonnegative real numbers $b\in Spec(\Delta_h)$ and $\phi\in Spec(\Delta_v)\subset Spec(S^1(\frac{\sqrt{2}}{2}))$, such that $\lambda=b+\phi$. As discussed at the first step, we see that $\bar{\lambda}:=b+\frac{1}{2}\phi\in Spec(S^3_B/\mathbb{Z}_2)$. According to Table 1, there are only three cases to be considered:

TABLE 1.

M	(μ_1, n_1)	(μ_2, n_2)	(μ_3, n_3)	$(\mu_k, n_k)(k > 1)$
$S^1(1)$	$(1, 2)$	$(4, 2)$	$(9, 2)$	$(k^2, 2)$
$S^1(\frac{\sqrt{2}}{2})$	$(2, 2)$	$(8, 2)$	$(18, 2)$	$(2k^2, 2)$
$S^2(\frac{\sqrt{2}}{2})$	$(4, 3)$	$(12, 5)$	$(24, 7)$	$(2k(k + 1), 2k + 1)$
$S^3(\sqrt{2})/\mathbb{Z}_2$	$(4, 9)$	$(12, 25)$	$(24, 49)$	$(2k(k + 1), (2k + 1)^2)$

(i) $\lambda = 0$. Obviously, in this case $\bar{\lambda} = 0$.

(ii) $\lambda \geqslant 12$. We claim that $\bar{\lambda} > 3$. Suppose $\bar{\lambda} \leqslant 3$. Then the inequality $\frac{1}{2}\phi \leqslant \bar{\lambda} \leqslant 3$ implies that $\phi = 0$ or 2. Hence $\bar{\lambda} \geqslant b = \lambda - \phi \geqslant 10$, which contradicts the assumption.

(iii) $\lambda = 4$. Clearly, $b, \phi \geqslant 0$ and $4 = b + \phi$. From Table 1, it follows that the possible values of ϕ are only 0 or 2. Let E_1 be the eigenspace corresponding to $\lambda = 4$. Again by Theorem 3.6 in [BB], there exist linearly independent functions $f_1, \cdots, f_9$ such that $E_1 = \mathrm{Span}\{f_1, \cdots, f_9\}$ and

$$\Delta_h f_k = b_k f_k, \ \Delta_v f_k = \phi_k f_k, \ b_k + \phi_k = 4, \quad \text{for } k = 1, 2, \cdots, 9.$$

Let i be the non-negative integer such that $\phi_k = 0$, for $k \leqslant i$; $\phi_k = 2$, for $k > i$. If $k \leqslant i$, the corresponding function f_k is induced from the base space. That is, there exists some function h_k such that $f_k = h_k \circ \pi$, $\Delta_B h_k = 4h_k$, where Δ_B is the Laplacian on the basic manifold $S^2(\frac{\sqrt{2}}{2})$. Since the multiplicity of $4 \in Spec(S^2(\frac{\sqrt{2}}{2}))$ is 3, it yields that $i = 3$. Namely, $\phi_k = 2$ and $b_k = 2$ for $k > 3$. Subsequently,

$$b_k + \frac{1}{2}\phi_k = 3 \in Spec(S^3_B/\mathbb{Z}_2).$$

Moreover, the space consisting of such functions has dimension 6.

Putting all these facts together, we complete the proof of the first part in Theorem 1.2.

5.2. **On the homogeneous focal submanifold M_1 with $(g, m_1, m_2) = (4, 4, 3)$.**

The last subsection will be devoted to calculating the first eigenvalue of the focal submanifold M_1 with dimension 10 and $(g, m_1, m_2) = (4, 4, 3)$ of OT-FKM type in $S^{15}(1)$. We use analogous method as that in Subsection 4.2 to define $M_1(\varepsilon)$ and $\widetilde{\Phi_\varepsilon}$. In the following, we just intend to calculate $\|\nabla\widetilde{\Phi}_\varepsilon\|_2^2$.

Firstly, let us make some notations. For any $x \in M_1$, denote simply $\nabla\varphi|_x =: X \in T_xM_1$. To simplify the illustration, we assume temporarily $|X| = 1$. For any point $a = (a_0, \cdots, a_4)$ in the unit sphere $S^4(1)$, let $P_a =: \sum_{\beta=0}^4 a_\beta P_\beta$ be an element in the Clifford sphere $\Sigma =: \Sigma(P_0, \cdots, P_4)$ spanned by $P_0, \cdots, P_4$. Denote $\xi_a =: P_a x$, then its shape operator is $A_{\xi_a} = \sum_{\beta=0}^4 a_\beta A_{\xi_\beta}$.

Next, in virtue of [FKM], for any $a \in S^4(1)$, we can decompose X with respect to eigenspaces of A_{ξ_a} into

$$X = Y_1 + Y + Y_{-1} \in E_1(A_{\xi_a}) \oplus E_0(A_{\xi_a}) \oplus E_{-1}(A_{\xi_a}).$$

Recall that $T_x^{\perp} M_1 = \mathrm{Span}\{P_\beta x \mid \beta = 0, \cdots, 4\}$ and $E_0(A_{\xi_a}) = \mathbb{R}\{QP_a x \mid Q \in \Sigma, \langle Q, P_a\rangle = 0\}$. Thus if we choose Q_j $(j = 1, \cdots, 4)$ in such a way that they constitute with P_a an orthonormal basis of Σ, then $Y = \sum_{j=1}^4 \langle X, Q_j P_a x\rangle Q_j P_a x$, and hence

$$|Y|^2 = \sum_{j=1}^{4} \langle X, Q_j P_a x\rangle^2 = \sum_{j=1}^{4} \langle P_a X, Q_j x\rangle^2 = |(P_a X)^{\perp}|^2. \tag{35}$$

Therefore, combining with the formula $A_{\xi_a} X = -(P_a X)^T$, we get $|Y|^2 = 1 - |A_{\xi_a} X|^2$.

On the other hand, notice that

$$|A_{\xi_a} X|^2 = \sum_{\alpha,\beta=0}^{4} a_\alpha a_\beta \langle A_{\xi_\alpha} X, A_{\xi_\beta} X\rangle = \sum_{\beta=0}^{4} a_\beta^2 |A_{\xi_\beta} X|^2 + T,$$

where T is the item consisting of the products $a_\alpha a_\beta \langle A_{\xi_\alpha} X, A_{\xi_\beta} X\rangle$ $(\alpha \neq \beta)$, whose integral on $S^4(1)$ vanishes since $\int_{S^4(1)} a_\alpha a_\beta dv = 0$ for $\alpha \neq \beta$. By the decomposition

$$P_\beta X = (P_\beta X)^T + (P_\beta X)^{\perp} = (P_\beta X)^T + \sum_{\gamma=0}^{4} \langle P_\beta X, P_\gamma x\rangle P_\gamma x,$$

we obtain that $1 = |A_{\xi_\beta} X|^2 + \sum_{\gamma=0}^4 \langle P_\beta X, P_\gamma x\rangle^2$. Therefore, the arguments above imply that

$$|Y|^2 = 1 - \sum_{\beta=0}^{4} a_\beta^2 |A_{\xi_\beta} X|^2 - T = 1 - \sum_{\beta=0}^{4} a_\beta^2 \Big(1 - \sum_{\gamma=0}^{4} \langle P_\beta X, P_\gamma x\rangle^2\Big) - T = \sum_{\beta,\gamma=0}^{4} a_\beta^2 \langle P_\beta X, P_\gamma x\rangle^2 - T,$$

which leads to

$$\begin{aligned}\int_{S^4(1)} |Y|^2 dv &= \int_{S^4(1)} \sum_{\beta,\gamma=0}^{4} a_\beta^2 \langle P_\beta X, P_\gamma x\rangle^2 dv = \frac{1}{5}\mathrm{Vol}(\mathrm{S}^4(1)) \cdot \sum_{\beta,\gamma=0}^{4} \langle X, P_\beta P_\gamma x\rangle^2 \\ &= \frac{2}{5}\mathrm{Vol}(\mathrm{S}^4(1)),\end{aligned}$$

where the last equality is followed from a crucial assertion that

$$T_x M_1 = \mathrm{Span}\{P_\beta P_\gamma x \mid \beta, \gamma = 0, \cdots, 4,\ \beta < \gamma\}$$

which holds only for homogeneous case with $(g, m_1, m_2) = (4, 4, 3)$(*cf.* Subsection 3.2.1 1), the case $q = 2$ in [QTY]). Subsequently, it is easily seen that

$$\int_{S^4(1)} |Y_1|^2 dv = \int_{S^4(1)} |Y_{-1}|^2 dv = \frac{1}{2} \int_{S^4(1)} (|X|^2 - |Y|^2) dv = \frac{3}{10}\mathrm{Vol}(\mathrm{S}^4(1)).$$

In this way, we obtain that

$$\begin{aligned}
\lim_{\varepsilon\to 0}\|\nabla\widetilde{\Phi}_\varepsilon\|_2^2 &= \int_0^{\frac{\pi}{4}}\int_{M_\theta}|\nabla\varphi(\phi_\theta)|^2\ dM_\theta d\theta\\
&= \int_0^{\frac{\pi}{4}}\int_{M_1}\int_{S^4(\sin\theta)}\frac{1}{\widetilde{k}_1^3\widetilde{k}_2^4\widetilde{k}_3^3}(\widetilde{k}_1^2|Y_1|^2+\widetilde{k}_2^2|Y|^2+\widetilde{k}_3^2|Y_{-1}|^2)\ dS^4(\sin\theta)dM_1d\theta\\
&= \int_0^{\frac{\pi}{4}}\int_{M_1}\mathrm{Vol}(\mathrm{S}^4(\sin\theta))|\nabla\varphi|^2\Big(\frac{3}{10}(\frac{1}{(\cos\theta+\sin\theta)^2}+\frac{1}{(\cos\theta-\sin\theta)^2})\\
&\quad+\frac{2}{5}\frac{1}{\cos^2\theta}\Big)\cdot\cos^3 2\theta\cos^4\theta\ dM_1d\theta\\
&= \|\nabla\varphi\|_2^2\cdot\mathrm{Vol}(\mathrm{S}^4(1))\Big(\frac{17}{2400}-\frac{\pi}{1280}\Big).
\end{aligned}$$

Combining with

$$\begin{aligned}
\lim_{\varepsilon\to 0}\|\widetilde{\Phi}_\varepsilon\|_2^2 &= \int_0^{\frac{\pi}{4}}\frac{1}{\widetilde{k}_1^3\widetilde{k}_2^4\widetilde{k}_3^3}\int_{M_1}\int_{S^4(\sin\theta)}\varphi(\phi_\theta)^2dS^4(\sin\theta)dM_1d\theta\\
&= \|\varphi\|_2^2\cdot\int_0^{\frac{\pi}{4}}\frac{1}{\widetilde{k}_1^3\widetilde{k}_2^4\widetilde{k}_3^3}Vol(S^4(\sin\theta))\ d\theta\\
&= \|\varphi\|_2^2\cdot\mathrm{Vol}(\mathrm{S}^4(1))\frac{1}{64}\cdot B(\frac{5}{2},2)\\
&= \|\varphi\|_2^2\cdot\mathrm{Vol}(\mathrm{S}^4(1))\frac{1}{560},
\end{aligned}$$

we conclude that

$$\lim_{\varepsilon\to 0}\frac{\|\nabla\widetilde{\Phi}_\varepsilon\|_2^2}{\|\widetilde{\Phi}_\varepsilon\|_2^2}=\frac{\|\nabla\varphi\|_2^2}{\|\varphi\|_2^2}\cdot(\frac{119}{30}-\frac{7\pi}{16})$$

Analogously as in Section 4,

$$\lambda_k(S^{15}(1))\leqslant\lambda_k(M_1)\cdot(\frac{119}{30}-\frac{7\pi}{16}).$$

In particular,

$$\lambda_{17}(M_1)\geqslant\frac{32}{\frac{119}{30}-\frac{7\pi}{16}}>12.$$

Finally, combing with Lemma 3.1 in [TY], we conclude that

$$\lambda_1(M_1)=\dim M_1=10,\ \text{with multiplicity } 16,$$

as required.

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School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, China

E-mail address: `zztang@bnu.edu.cn`

Department of Mathematics, Hangzhou Normal University, Zhejiang 310036, China

E-mail address: `yuqxie@pku.edu.cn`

School of Mathematical Sciences, Laboratory of Mathematics and Complex Systems, Beijing Normal University, Beijing 100875, China

E-mail address: `wjyan@bnu.edu.cn`